An Energy Approach for Orbital Transfers

D.H. May*
6555 Aerospace Test Group, Cape Canaveral Air Force Station, Florida

Evaluating the energy change in an orbital transfer vehicle due to its propulsion system provides an approach for solving a variety of transfer problems and expressing results in common propulsion terms. A high-thrust geosynchronous mission is given as an example to demonstrate one application and for comparing low-thrust results. Solutions to low-thrust ion engine transfers are developed by equating work performed to total energy change. Low-thrust plane changes are also analyzed. Using the energy approach, orbital transfer problems are analogous to certain classical problems in mechanics.

Nomenclature

E	= total mechanical energy
$\mathcal{E}$	= specific mechanical energy
F	= force
g_c	= Earth surface gravitational constant
I_{sp}	= specific impulse
KE	= total kinetic energy
m	= mass
$\dot{m}$	= mass flow rate
M	= total vehicle mass
r	= radius
t	= time
$\mathfrak{J}$	= thrust
v	= velocity
$\mathcal{W}$	= work per unit mass
μ	= gravitational parameter for Earth
ν	= polar angle of circular orbit
ν'	= small increment of polar angle
ψ	= inclination of orbit
ψ'	= incremental change in inclination

Subscripts

f	= final orbit
k	= kinetic (energy)
p	= propulsion system action
0	= initial, parking orbit
1	= first burn termination

Introduction

CHEMICAL propulsion systems in high-thrust upper-stage vehicles, such as the solid rocket motor inertial upper stage, produce an impulse that may be assumed instantaneous. In solving orbital transfer problems for such vehicles, velocity vector diagrams are constructed for each engine firing and the vehicle is assumed to make an instantaneous transition from one orbit to another.[1] Velocity vector diagrams can be constructed because only the kinetic energy of the vehicle changes at the instant of burn.

Classical problems in mechanics that involve a conversion of energy between kinetic and potential are solved most easily and directly using energy methods. An energy approach has similar advantages in solving the low-thrust orbital transfer problem. These transfers are of long duration, therefore, orbital motion during propulsion system action must be included in the solution. Energy conversion between kinetic and potential is accounted for when the solution is based upon total energy.

Rocket Energy Equation

A propulsion system imparts energy to its vehicle. An expression for the quantity of energy follows the development of the rocket equation for a change in velocity.[2] Vehicle velocity gain resulting from propulsion system operation may be expressed as

$$\int_0^{v_p} \mathrm{d}v = -I_{sp}g_c\int_{M_0}^{M}\frac{\mathrm{d}m}{m} = -I_{sp}g_c \ln\left(\frac{M_0-\dot{m}t}{M_0}\right) \qquad (1)$$

This velocity gain is the actual change of vehicle velocity if the energy change is kinetic only, as in high-thrust problems.

The rate of change of kinetic energy KE of any moving body is

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{KE}) = \frac{\mathrm{d}}{\mathrm{d}t}(\tfrac{1}{2}mv^2) = Fv - \tfrac{1}{2}\dot{m}v^2$$

where the force F is the rate of change of the momentum mv. A change in kinetic energy due to propulsion alone may be expressed as

$$\Delta\mathrm{KE} = \int Fv_p\mathrm{d}t - \int\frac{\dot{m}}{2}v_p^2\mathrm{d}t \qquad (2)$$

With the substitution for v_p from Eq. (1), Eq. (2) becomes

$$\Delta\mathrm{KE} = -\int \mathfrak{J}I_{sp}g_c\ln\left(\frac{M_0-\dot{m}t}{M_0}\right)\mathrm{d}t$$

$$-\tfrac{1}{2}\int\dot{m}\left[I_{sp}g_c\ln\left(\frac{M_0-\dot{m}t}{M_0}\right)\right]^2\mathrm{d}t$$

where the thrust $\mathfrak{J}$ is substituted for F. Recognizing that $\mathfrak{J}$ is equal to $\dot{m}I_{sp}$ and integrating

$$\Delta\mathrm{KE} = \tfrac{1}{2}I_{sp}^2g_c^2(M_0-\dot{m}t)\left[\ln\left(\frac{M_0}{M_0-\dot{m}t}\right)\right]^2$$

where total vehicle mass at any time is $M_0-\dot{m}t$. A rocket energy equation gives specific kinetic energy change or work as a function of mass flow rate and time,

$$\Delta\mathcal{E}_k = \mathcal{W} = \tfrac{1}{2}I_{sp}^2g_c^2\left[\ln\left(\frac{M_0}{M_0-\dot{m}t}\right)\right]^2 \qquad (3)$$

Presented as Paper 84-2054 at the AIAA/AAS Astrodynamics Conference, Seattle, WA, Aug. 20-22, 1984; submitted Jan. 4, 1985; revision received May 13, 1985.

*Lt. Col., USAF. Adjunct Faculty Member, Space Technology, Florida Institute of Technology, Melbourne, FL. Member AIAA.

Vector notation has been omitted in the equations presented here. However, applications must account for direction as well as magnitude. High-thrust problems are solved by vectorially adding propulsion velocity change to prior vehicle velocity. The thrust vector is constrained to a fixed direction.[1] Low-thrust problems may be solved with an appropriate constraint applied to the thrust vector, such as continuous alignment with vehicle velocity. The velocity term in Eq. (1) is integrated from zero to v_p but the velocity at $t=0$ is not ignored.

High-Thrust Transfers

Numerous references describe satellite motion about its parent body. The sum of specific kinetic and potential energies is expressed as

$$\mathcal{E}=\frac{v^2}{2}-\frac{\mu}{r}$$

This equation describes unpowered orbital motion or coasting between impulse events. Chemical propulsion systems in upper-stage vehicles typically complete a required burn in a time period that is very short compared to the time of orbital transfer, and the assumption that each burn gives an instantaneous impulse may be applied. The problem at each impulse is analogous to classical conservation of linear momentum problems.

Using a Hohmann transfer for low-Earth circular orbit to geosynchronous orbit as a representative problem, a precise energy change requirement may be determined for each of two burns. In compliance with Eq. (3), the change in specific total energy will equal the change in specific kinetic energy during each burn. A velocity vector addition problem results if a plane change is made as shown in Fig. 1.[1] The angle ψ' is the plane change during the burn. If the burn is over the equator, as in a typical transfer to geosynchronous orbit, ψ' is also orbit inclination change. Applying the law of cosines to this vector triangle and dividing by 2 yields an equation relating kinetic energies before and after the burn, i.e.,

$$\Delta\mathcal{E}_{k1}=\mathcal{E}_{k0}+\mathcal{E}_{k1}-2\sqrt{\mathcal{E}_{k0}\mathcal{E}_{k1}}\cos\psi' \qquad (4)$$

The change in mass or the propellant consumed during a burn may be found from Eq. (3) using $\Delta\mathcal{E}_k$ as determined by Eq. (4),

$$\Delta M=\dot{m}t=M_0\left[1-\exp\left(\frac{-\sqrt{2(\Delta\mathcal{E}_k)}}{I_{sp}g_c}\right)\right]$$

Low Thrust

Orbital Transfers

An orbital transfer vehicle with ion engines would present a useful application for Eq. (3). These relatively low-thrust engines would operate over long time periods imparting energy to the vehicle very gradually. Orbital motion for such a transfer would be a gradually expanding spiral. As the vehicle expands outward from an initially circular orbit, the path is often assumed to be a series of concentric circles with the velocity always equal to circular orbit velocity.[2,3] This assumption provides an important mathematical relationship since kinetic, potential, and total energies are each functions of radius.

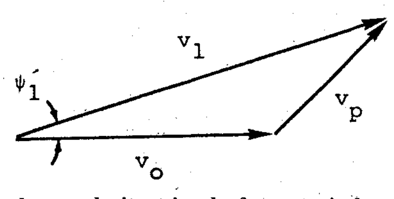

Fig. 1 First burn velocity triangle for a typical geosynchronous transfer.

Equation (3) determines the kinetic energy change during engine operation. In low-thrust, long-duration transfers, a conversion occurs between orbital kinetic and potential energies during the propulsion event. Therefore, a kinetic energy change approach must be modified to account for this conversion. A solution based upon work performed may be obtained provided that all of the propulsion system work equates to energy change. This imposes a constraint of continuous alignment of the thrust and velocity vectors (same direction for ascent, opposite direction for descent).

Continuous alignment of v_p with vehicle velocity permits a useful expression to be written concerning the energy states.

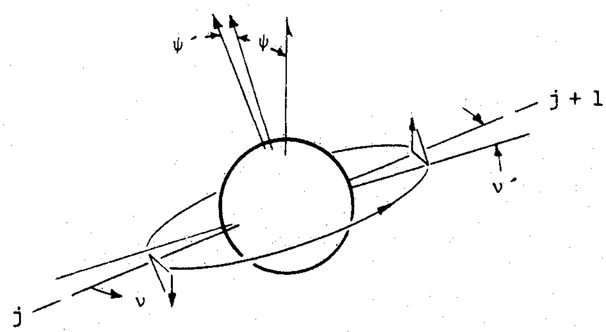

Fig. 2 Orbital sketch for low-thrust incremental plane changes.

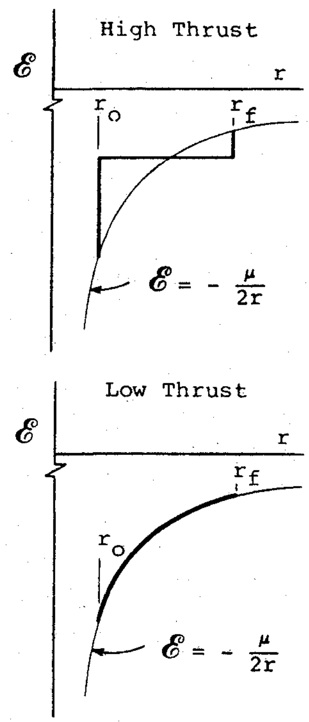

Fig. 3 Total energy vs radius for high- and low-thrust in-plane geosynchronous transfers.

The velocity triangle that represents the single burn reduces to linear addition of velocity vectors (as in Fig. 1) with ψ' equal to zero. An equation similar to Eq. (4) applies, but reduces to

$$\mathcal{W}=\Delta\mathcal{E}_k=\mathcal{E}_{k0}+\mathcal{E}_k-2\sqrt{\mathcal{E}_{k0}\mathcal{E}_k} \qquad (5)$$

where $\mathcal{E}_k$ is the kinetic energy associated with the final vector. In the special case of a low-thrust transfer where velocity is always equal to circular velocity, Eq. (5) applies. Total energy is equal in magnitude to kinetic energy at any point along the path from the initial to the final state, and gravity always acts normal to the velocity vector. Thus, an expression for work performed between the initial circular orbit and any subsequent circular orbit may be written by expressing kinetic enregy as a function of radius,

$$\mathcal{W} = \frac{\mu}{2}\left(\frac{1}{\sqrt{r_0}} - \frac{1}{\sqrt{r}}\right)^2 \tag{6}$$

The work expressed in Eq. (3) equates to the work requirement of Eq. (6) for a low-thrust transfer. With roots selected to give positive values, this becomes

$$I_{sp} g_c \ln\left(\frac{M_0}{M_0 - \dot{m}t}\right) = \sqrt{\mu}\left(\frac{1}{\sqrt{r_0}} - \frac{1}{\sqrt{r}}\right)$$

The mass at any time may be expressed as

$$M = M_0 - \dot{m}t = M_0 \exp\left[\frac{-\sqrt{\mu}}{I_{sp} g_c}\left(\frac{1}{\sqrt{r_0}} - \frac{1}{\sqrt{r}}\right)\right] \tag{7}$$

and transfer time to any orbit from r_0 is given by

$$t = \frac{M_0}{\dot{m}}\left\{1 - \exp\left[\frac{-\sqrt{\mu}}{I_{sp} g_c}\left(\frac{1}{\sqrt{r_0}} - \frac{1}{\sqrt{r}}\right)\right]\right\} \tag{8}$$

Plane Changes

Low-thrust plane changes may be accomplished by incrementally precessing orbit inclination as the vehicle crosses the equatorial plane. If the thrust vector is turned perpendicular to the plane of the orbit at each nodal crossing, then the torque acting on the angular momentum vector will result in a small plane change. Also, if the arc through which the vehicle passes during this thrusting (ν') is small, then the incremental precession will very nearly equal the incremental plane change ψ' at each node. Figure 2 shows a sketch of a single revolution with plane change impulses at the nodes. To reduce inclination, the vehicle is thrusted toward the northern hemisphere at the descending node and in the opposite direction at the ascending node. For simplicity only a thrust vector perpendicular to the orbit plane is considered, but the appropriate vector component could also be analyzed.

Precession of the angular momentum vector is given by[4]

$$\dot{\psi} = \frac{\text{Torque}}{\text{Angular momentum}} = \frac{\mathfrak{T}r}{rvm}$$

or

$$\dot{\psi} = \frac{I_{sp} g_c \dot{m}}{m\sqrt{\mu/r}}$$

for circular orbits. This becomes

$$\int d\psi = \frac{I_{sp} g_c}{\sqrt{\mu/r}} \int \frac{dm}{m}$$

The incremental angle change in degrees is

$$\psi' = \frac{\pi}{180} I_{sp} g_c \sqrt{\frac{r_j}{\mu}} \ln\left(\frac{M_j}{M_j - \dot{m}t_j}\right) \tag{9}$$

where M_j is the mass of the vehicle at the time of the event and t_j is the duration of the plane change thrusting. This time for a single plane change event is a fraction of the orbital period as determined by the angle ν'. The period of a circular orbit is given by[5]

$$\text{Period (circular)} = \frac{2\pi}{\sqrt{\mu}} r^{3/2}$$

Therefore,

$$t_j = \frac{\nu'}{180} \frac{\pi}{\sqrt{\mu}} r^{3/2}$$

with ν' in degrees. This value for event time may be substituted into Eq. (9).

Using Eq. (3) for work performed, the work required for each plane change event is

$$\mathcal{W}_j = \frac{1}{2}\left(\frac{180}{\pi}\right)^2 \frac{\mu}{r_j} (\psi_j')^2$$

Equation (9) will give increments of plane change at the nodes where the thrust is aligned as described. Vehicle mass for each of these events M_j must be found by an accounting of mass changes in all prior radius change and plane change thrusting. Total elapsed time will be given by Eq. (8) plus the sum of all t_j increments. Mass change is the product of this elapsed time and $\dot{m}$.

The plane change angle is a function of r and increases at greater radii for the same ν'. Therefore, it is advantageous to make plane changes at the maximum transfer radius. However, ν' must be small and this may require the energy expenditure for large plane changes to be allocated over a range of radii.

It should be noted that a maneuver similar to the plane change described above may be useful to rotate or shift the node line. If the thrust vector is pointed perpendicular to the orbit plane at a point near maximum latitude, then orbit precession will result in a shift in the longitude of the ascending node. The angle through which it shifts may be determined by using Eq. (9) and solving a spherical triangle problem.

Discussion and Summary

At any point in its orbit, a satellite has a specific energy state as determined by its instantaneous radius and velocity. Work may be performed to change the energy state by propulsion system action, and a new energy state may be described at the time of thrust termination. For very brief thrusting the energy change can be assumed to be kinetic only, but for long-duration firings the energy state achieved is a result of the combination of propulsion system action and orbital motion. Initial and final energy states for a propulsive event are independent of the path taken, but work performed is path-dependent as in classical problems.

The two distinctly different problems presented for a transfer between low-Earth and geosynchronous orbit demonstrate different work requirements. Path dependency of the work required is shown by plotting $\mathcal{E}$ vs r, as in Fig. 3, for each transfer. The high-thrust plot shows two kinetic energy changes (engine burns) as vertical lines, and the horizontal line is a coast in an elliptical orbit with total energy remaining constant. This high-thrust path minimizes the work required for the transfer. For purposes of comparing transfers, no plane change is considered in either case in Fig. 3.

The simplifying assumption that a low-thrust vehicle maintains a velocity that always equals circular orbit velocity dictates the path for the low-thrust curve in Fig. 3. Equation (6) gives work required to move along the curved $\mathcal{E}$ vs r path. Each pound of total vehicle weight requires more work via this path than in the two-burn, high-thrust transfer.

Another interesting comparison of these two separate problems is shown by plotting vehicle mass at the final energy state as a fraction of initial mass. Figure 4 shows these curves on the same logarithmic scale. Reasonable ranges of I_{sp} values for each transfer are different. Chemical systems used for high-thrust transfers cover a range approximated by the solid line; selection of reasonable ion engine values for low-thrust transfers produces the dashed line.[2] Mass remaining at the end of the mission includes vehicle dry weight and payload. The fraction above the line gives propellant required.

The advantage of a high specific impulse is apparent in Fig. 4. If transfer time were not a consideration, low-thrust systems would clearly be superior to high-thrust upper stages. Mass fractions as determined by high-thrust transfers are plotted as a dotted line extending into the region of high specific

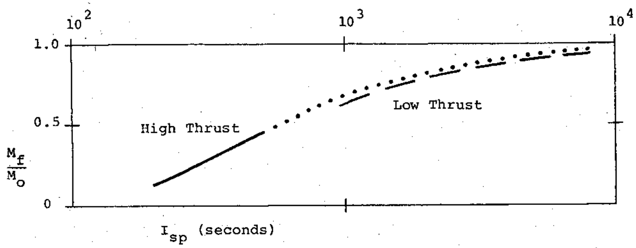

Fig. 4 Mass ratio vs specific impulse for the in-plane geosynchronous mission.

impulse values. Although this may represent impractical performance, it serves to show the upper limit for any conceivable system with high I_{sp}. On the other hand, the dashed line represents a conservative result. A system that would truly satisfy the assumption of continuous circular orbital velocity would fail to expand in a spiraling orbit as the radius increases. Since vehicle mass decreases during the transfer and it would be reasonable to maintain a constant thrust, expansion of the spiral would be expected. Thus, it is feasible that a low-thrust transfer could be performed with less propellant consumed than Eq. (7) would indicate and in less time than Eq. (8) yields. A practical ion engine system serving as an orbital transfer vehicle would have a performance that lies between the dashed and dotted lines of Fig. 4.

Low-thrust orbital transfers were studied by Wiesel and Alfano,[3] who employed a control Hamiltonian and Lagrange multipliers. The similarities in deduced conclusions between the energy approach and their approach are interesting. In their analysis, Wiesel and Alfano introduced an independent variable that represents a form of the rocket equation and was described as a total accumulated velocity change for the vehicle. This parameter appears to be a variation of the kinetic energy change or work performed. One conclusion stated was that mass flow rate need not remain constant to determine transfer time. Equation (8) supports the same conclusion, provided the assumption that vehicle velocity always equals circular orbit velocity remains valid. A change in $\dot{m}$ would only require the appropriate adjustment in time to achieve the necessary energy change.

Wiesel and Alfano also demonstrated by example a greater accumulated velocity (energy) for a low-thrust transfer than for a Hohmann transfer between the same two circular orbits. This difference is apparent in the different work requirements found for the high- and low-thrust transfers. Another conclusion that is corroborated is the greater efficiency of making plane changes at greater radii.

Conclusion

An energy approach to analyzing orbital transfers provides an understanding of these problems which parallels energy approaches to classical problems in mechanics. Although the equations derived for low-thrust problems are exact, they rely on an idealized circular orbit velocity assumption. With proper representation of the expanding spiral, and energy path may be mathematically described and the energy approach can yield practical solutions.

Expressing work performed in terms of common propulsion system parameters has distinct advantages for applying industry standards and methods to orbital problems. Specific impulse and mass flow rate are values that can be established by testing and are thoroughly understood. Evaluating vehicle performance in terms of these parameters is consistent with current practices.

References

[1]Chu, S.T., Lang, T.J., and Winn, E.E., "A Velocity Matching Technique for Three-Dimensional Orbit Transfer in Conceptual Mission Design," *Journal of the Astronautical Sciences,* Vol. XXVI, Oct.-Dec. 1978, pp. 343-368.

[2]Hill, P.G. and Peterson, C.R., *Mechanics and Thermodynamics of Propulsion,* Addison-Wesley, Reading, MA, 1970.

[3]Wiesel, W.E. and Alfano, S., "Optimal Many-Revolution Orbit Transfer," AAS/AIAA Paper 83-352, Aug. 1983.

[4]Richards, J.A. et al., *Modern University Physics,* Addison-Wesley, Reading, MA, 1960.

[5]Bate, R.R., Mueller, D.D., and White, J.E., *Fundamentals of Astrodynamics,* Dover Publications, New York, 1971.